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# SPECTRA OF POLAR FACTORS OF HYPONORMAL OPERATORS(1)

BY

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ABSTRACT. An investigation is made of the interdependence and properties of the spectrum of a hyponormal operator $T$ and of the spectra, and absolutely continuous spectra, of the factors in a polar factorization of $T$ when the latter exists.

**1. Introduction.** Only bounded operators on a fixed separable Hilbert space $H$ will be considered in this paper. An operator $T$ will be said to have a polar factorization if $T = UP$ where $U$ is unitary and $P$ is a nonnegative selfadjoint operator. (Other factorizations in which $U$ is not unitary but is only an isometry or a partial isometry, cf. Halmos [1, p. 68], or Kato [3, p. 334], will not be considered.) Thus, if $T$ has a polar factorization $T = UP$, then $T^* = PU^*$ and $T^*T = P^2$, hence $P = (T^*T)^{1/2}$, so that

$$T = UP, \qquad U \text{ unitary and } P = (T^*T)^{1/2}. \tag{1.1}$$

In general the unitary factor is not unique. In case $T$ is nonsingular, that is, if 0 is not in its spectrum, the polar factorization exists, is unique, and was given by Wintner [12]; a generalization was obtained by von Neumann [4, p. 307].

As noted above, if $T = UP$ where $U$ is unitary and $P$ is nonnegative then necessarily $P = (T^*T)^{1/2}$. Also,

$$TT^* = U(T^*T)U^* \text{ (equivalently, } (TT^*)^{1/2} = U(T^*T)^{1/2}U^*), \ U \text{ unitary.} \tag{1.2}$$

Conversely, it was shown by Hartman [2], using the above mentioned result of von Neumann, that if $T$ is arbitrary then the nonzero spectra of $T^*T$ and $TT^*$ are identical, including multiplicities of both point and continuous spectra, while 0 may occur in the point spectra of $T^*T$ and $TT^*$ with different multiplicities. Further, (1.2) holds for some unitary $U$ if and only if the multiplicities of 0 in the point spectra of $T^*T$ and $TT^*$ (equivalently, of $T$ and of $T^*$) are equal, that is,

$$\dim\{x: Tx = 0\} = \dim\{x: T^*x = 0\}. \tag{1.3}$$

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In addition (cf. [2, p. 234], $T$ has a polar factorization (1.1), for some unitary $U$, if and only if (1.2) holds for some (not necessarily the same) unitary $U$, or, equivalently, if and only if (1.3) holds. In this case, the unitary operator $U$ of (1.1) (but, of course, not that of (1.2)) is uniquely determined if 0 is not in the point spectrum of $T$ (and/or $T^*$), that is, if the common dimension of (1.3) is 0.

Next, an operator is said to be hyponormal if

$$T^*T - TT^* = D \geq 0, \tag{1.4}$$

and completely hyponormal if, in addition, there is no nontrivial subspace reducing $T$ on which $T$ is normal. It was shown in Putnam [8] that if $T$ is completely hyponormal then its spectrum, $\mathrm{sp}(T)$, has positive planar measure and, in fact,

(1.5) if $T$ is completely hyponormal then $\mathrm{meas}_2(\mathrm{sp}(T) \cap \alpha) > 0$ whenever $\mathrm{sp}(T) \cap \alpha \neq$ empty set,

where $\alpha$ denotes any open disk of the complex plane.

Let $T_z = T - zI$ for any complex $z$. Then $T_z^* T_z - T_z T_z^* = T^*T - TT^*$ and hence

$$\{x\colon T_z x = 0\} \subset \{x\colon T_z^* x = 0\} \quad \text{if } T \text{ is hyponormal.} \tag{1.6}$$

Hence, if $z$ is in the point spectrum of a hyponormal $T$ the corresponding eigenspace is a reducing space of $T$ on which it is normal. It is also clear that if $T$ is hyponormal and if 0 is not in the point spectrum of $T^*$ then $T$ has a (unique) polar factorization (1.1). Of course, if $T$ is normal, and whether or not 0 is in the point spectrum of $T^*$, equality holds in (1.6) for all $z$, in particular, for $z = 0$, and it follows that $T$ must have a (that is, at least one) polar factorization. Such a factorization is easily constructed, for instance, from the spectral resolution of the operator. The unilateral shift (cf. Halmos [1, p. 40]) is an example of a completely hyponormal operator which fails to have a polar factorization (1.1).

Recall that $A$ is a selfadjoint operator with the spectral resolution $A = \int t\, dE_t$ then the set $H_a(A)$ of elements $x$ in $H$ for which $\|E_t x\|^2$ is an absolutely continuous function of $t$ is a subspace of $H$ reducing $A$. The operator $A$ is said to be absolutely continuous if $H_a(A) = H$. Similar concepts can be defined for a unitary operator $U = \int_0^{2\pi} e^{it}\, dE_t$; cf. [6, p. 19].

If $T$ is hyponormal with the rectangular representation $T = A + iB$ ($A$, $B$ selfadjoint) it was shown in Putnam [5] (cf. also [6, p. 46]) that, exactly as in the case when $T$ is normal, the spectra of $A$ and $B$ are precisely the projections, as real sets, of the spectrum of $T$ onto the real and imaginary axes. Further (cf. [6, pp. 42–43]), both $H_a(A)$ and $H_a(B)$ contain the least subspace of $H$ reducing $T$ and containing the range of $T^*T - TT^*$. In particular, if $T$ is completely hyponormal, $A$ and $B$ are absolutely continuous. This paper will deal with an analogous investigation of the spectrum of $T$ and of the spectra, and absolutely continuous spectra, of the components of a polar factorization of $T$, when the latter exists.

For use below, recall that a number $t$ is said to be in the essential spectrum of a selfadjoint operator $A$, $\mathrm{essp}(A)$, if $t$ is either a limit point of $\mathrm{sp}(A)$ or is an eigenvalue of infinite multiplicity. The point spectrum of any operator $T$ will be denoted by $\mathrm{ptsp}(T)$.

2. **Theorem 1.** *Let* $T$ *be hyponormal and let*

(2.1) $$z \in \text{boundary of } \mathrm{sp}(T).$$

*Then*

(2.2) $$|z| \in \mathrm{sp}(T^*T)^{1/2} \cap \mathrm{sp}(TT^*)^{1/2}.$$

*Further, if* $T$ *is completely hyponormal, then*

(2.3) $$|z| \in \mathrm{essp}(T^*T)^{1/2} \cap \mathrm{essp}(TT^*)^{1/2}.$$

**Proof.** The hypothesis (2.1) implies that there exists a sequence of unit vectors, $\{x_n\}$, for which $(T - zI)x_n \to 0$. Since $T$ is hyponormal, also $(T^* - \bar{z}I)x_n \to 0$ and so $(T^*T - |z|^2 I)x_n \to 0$ and $(TT^* - |z|^2 I)x_n \to 0$, hence also $((T^*T)^{1/2} - |z|I)x_n \to 0$ and $((TT^*)^{1/2} - |z|I)x_n \to 0$, and so (2.2) follows. Further, if $z$ is an isolated point of $\mathrm{sp}(T)$, $T$ has a normal part with eigenvalue $z$ (cf. Stampfli [11, p. 473] or Putnam [8]). Hence, if $T$ is completely hyponormal, it follows from [7, Theorem 2 of p. 506], that the above sequence $\{x_n\}$ can be chosen so as to converge weakly to 0, and hence (2.3) holds.

**Remarks.** The above argument shows that if $T$ is normal, then (2.2) holds, if, instead of (2.1), it is supposed only that

(2.4) $$z \in \mathrm{sp}(T).$$

In general, however, if $T$ is only hyponormal, condition (2.4) does not imply (2.2). One need only let $T$ denote the unilateral shift operator, so that on the $l^2$ sequence space $x = (x_1, x_2, \cdots)$, $Tx = (0, x_1, x_2, \cdots)$. Then $\mathrm{sp}(T)$ is the closed unit disk but $T^*T = I$ and $TT^* = \mathrm{diag}(0, 1, 1, \cdots)$.

As noted earlier, the unilateral shift fails to have a polar factorization (1.1). However, even if $T$ is hyponormal and nonsingular, in which case a polar factorization (1.1) is assured, still (2.4) does not imply (2.2). To see this, consider the doubly infinite nonnegative diagonal matrices

$$A = \mathrm{diag}(\cdots, a_{-1}, a_0, a_1, \cdots) \quad \text{and} \quad B = \mathrm{diag}(\cdots, b_{-1}, b_0, b_1, \cdots)$$

with $a_i = 4$ for $i \geq 1$, $a_i = 1$ for $i \leq 0$, $b_i = 4$ for $i \geq 0$, $b_i = 1$ for $i \leq -1$. Let $P$ denote the nonnegative square root of $B$ and put $T = UP$, where $U$ is the unitary bilateral shift on the sequence space of vectors $x = (\cdots, x_{-1}, x_0, x_1, \cdots)$, $\Sigma |x_i|^2 < \infty$, defined by $(Ux)_n = x_{n-1}$ $(n = 0, \pm 1, \pm 2, \cdots)$. Since $A = UBU^*$ then

$$T^*T - TT^* = B - A = \operatorname{diag}(\cdots, d_{-1}, d_0, d_1, \cdots)$$

with $d_0 = 3$ and $d_i = 0$ for $i \neq 0$. Thus $T$ is hyponormal but not normal. Also, $\operatorname{sp}(T) = \{z: 1 \leq |z| \leq 2\}$, as can be deduced, for instance, from the results of this paper (cf. Theorems 8, 9 below). However, $\operatorname{sp}(T^*T)^{1/2} = \operatorname{sp}(TT^*)^{1/2} = \{1, 2\}$.

**Theorem 2.** *Let $T$ be hyponormal and suppose that $z \in \operatorname{sp}(T)$ and $\bar{z} \notin$ $\operatorname{ptsp}(T^*)$. Then $|z| \in \operatorname{essp}(T^*T)^{1/2} \cap \operatorname{essp}(TT^*)^{1/2}$.*

**Proof.** Since $T$ is hyponormal, $T_z^* T_z \geq T_z T_z^*$, where $T_z = T - zI$, and so, since $z \in \operatorname{sp}(T)$, $0 \in \operatorname{sp}(T_z T_z^*)$. Also, since $\bar{z} \notin \operatorname{ptsp}(T^*)$, then $z \notin \operatorname{ptsp}(T)$. Consequently, 0 is in the essential spectra of both $T_z^* T_z$ and $T_z T_z^*$. In view of the inequality $T_z^* T_z \geq T_z T_z^*$, there exists a sequence of unit vectors, $\{x_n\}$, converging weakly to 0 for which both $(T - zI)x_n \to 0$ and $(T^* - \bar{z}I)x_n \to 0$ and hence $(T^*T - |z|^2 I)x_n \to 0$ and $(TT^* - |z|^2 I)x_n \to 0$. Thus, $|z|^2$ is in the essential spectra of both $T^*T$ and $TT^*$, and the assertion of the theorem follows.

3. **Theorem 3.** *Let $T$ be hyponormal with a polar factorization* (1.1). *Suppose that $z \neq 0$ and satisfies* (2.4) *and that $z = |z|e^{i\theta}$. Then, for any $U$ of* (1.1),

$$e^{i\theta} \in \operatorname{sp}(U). \tag{3.1}$$

**Proof.** Let $z_1 = re^{i\theta}$ where $r = \max\{|z|: z = |z|e^{i\theta}$ and $z \in \operatorname{sp}(T)\}$ (hence $r > 0$). Clearly, $z_1$ is a boundary point of $\operatorname{sp}(T)$ and, as in Theorem 1, there exists a sequence of unit vectors, $\{x_n\}$, such that $(T - z_1 I)x_n \to 0$ and $(T^* - \bar{z}_1 I)x_n \to 0$ and hence also $((T^*T)^{1/2} - rI)x_n \to 0$. But $(T - z_1 I)x_n = U(T^*T)^{1/2}x_n - z_1 x_n \to 0$. Since $r > 0$, this implies that $(Ux_n - e^{i\theta}x_n) \to 0$ and, hence, that (3.1) holds.

4. **Theorem 4.** *Let $T$ be hyponormal and nonsingular, so that $T$ has a (unique) polar factorization* (1.1). *Then if $e^{i\theta} \in \operatorname{sp}(U)$, there exists a $z = |z|e^{i\theta} \neq 0$ satisfying* (2.4).

**Proof.** We have $T = UP$ and

$$P^2 - UP^2U^* = T^*T - TT^* = D \geq 0. \tag{4.1}$$

Since $e^{i\theta} \in \operatorname{sp}(U)$ there exists a sequence of unit vectors, $\{x_n\}$, satisfying $(U - e^{i\theta}I)x_n \to 0$, hence $(U^* - e^{-i\theta}I)x_n \to 0$. Clearly,

$$\|D^{1/2}x_n\|^2 = (Dx_n, x_n) = (P^2x_n, x_n) - (UP^2U^*x_n, x_n)$$

$$= (P^2x_n, x_n) - (P^2U^*x_n, U^*x_n) \to 0,$$

and so $Dx_n \to 0$. Hence, by (4.1), $P^2x_n - UP^2U^*x_n \to 0$, that is, $(U^* - e^{-i\theta}I)P^2x_n \to 0$. A similar argument shows that $(U^* - e^{-i\theta}I)f(P^2)x_n \to 0$, where $f(t)$ is a polynomial, or, via the functional calculus, a continuous function on $(-\infty, \infty)$. It then follows (cf. a similar argument in [6, p. 46]) that there exists a number $s > 0$ and a sequence of unit vectors $\{y_n\}$ such that $(P^2 - sI)y_n \to 0$ and $(U^* - e^{-i\theta}I)\, y_n \to 0$, hence also $(P - s^{1/2}I)y_n \to 0$ and $(U - e^{i\theta}I)y_n \to 0$. Consequently, if $z = s^{1/2}e^{i\theta}$, then $(T - zI)y_n \to 0$ (also $(T^* - \bar{z}I)y_n \to 0$) and so (2.4) holds.

5. **Theorem 5.** *Let $T$ be hyponormal with a polar factorization* (1.1) *and suppose that*

$$(5.1) \qquad 0 \notin \operatorname{ptsp}(T).$$

*If $e^{i\theta} \in \operatorname{sp}(U)$ then there exist $z_n = |z_n|e^{i\theta_n} \neq 0$, $z \in \operatorname{sp}(T)$, for which $\theta_n \to \theta$.*

**Remark.** Note that if $T$ is completely hyponormal then the hypotheses (1.1) and (5.1) are certainly fulfilled.

**Proof.** In case $T$ is nonsingular the above theorem follows from Theorem 4. The theorem also is clear if $T$ is singular and if there does not exist some open wedge

$$(5.2) \qquad W = \{z\colon z = re^{it},\ r > 0,\ a < t < b\}, \qquad a < \theta < b,$$

for which

$$(5.3) \qquad \operatorname{sp}(T) \cap W \text{ is empty.}$$

Consequently, it is sufficient to show that if $T$ is singular then the assumption that there exists a wedge $W$ of (5.2) satisfying (5.3) leads to a contradiction.

Suppose then the existence of such a wedge. Consider the bisector of $W$, that is, the half-line $\{z\colon z = re^{i\frac{1}{2}(a+b)},\ r > 0\}$ and choose complex numbers $s_n = |s_n|e^{i\frac{1}{2}(a+b)} \neq 0$ on this half-line satisfying $s_n \to 0$. It is clear that each (hyponormal) operator $T_n = T - s_nI$ is nonsingular and that, by (5.3), $\operatorname{sp}(T_n) \cap W$ is empty. If $T_n = U_nP_n$ is the (unique) polar factorization of $T_n$ then, by Theorem 4,

$$(5.4) \qquad e^{it} \in \operatorname{sp}(U_n) \quad (n = 1, 2, \cdots) \text{ whenever } a < t < b.$$

But $\|T - T_n\| \to 0$ and hence $\|P - P_n\| = \|(T^*T)^{1/2} - (T_n^*T_n)^{1/2}\| \to 0$. Also, $U_nP - UP = T_n - T + U_n(P - P_n)$, so that $\|U_nP - UP\| \to 0$ and, in particular, $U_nPx \to UPx$ (strongly) for all $x$ in $H$. In view of (5.1), $0 \notin$ point spectrum of $P = (T^*T)^{1/2}$, hence the range of $P$ is dense, and consequently

$$(5.5) \qquad U_n \to U \quad \text{(strongly).}$$

By (5.4), $\|(U_n - e^{i\theta}I)x\| \geq c\|x\|$ for all $x$ where $c$ is some positive constant, and hence, by (5.5), $\|(U - e^{i\theta}I)x\| \geq c\|x\|$. This implies that $e^{i\theta} \notin \mathrm{sp}(U)$, a contradiction, and the proof of Theorem 5 is complete.

6. **Theorem 6.** *Let $T$ be completely hyponormal and have a polar factorization $T = UP$ of* (1.1). *In addition, suppose that there exists some open wedge $W$ of* (5.2) *satisfying* (5.3). *Then*

(6.1) $P = (T^*T)^{1/2}$ *(hence also* $(TT^*)^{1/2}$*) and $U$ are absolutely continuous.*

**Proof.** It follows from Theorem 4 that no $e^{i\theta}$, $a < \theta < b$, can belong to $\mathrm{sp}(U)$. It now follows from (4.1) and the theorem of [6, p. 21], that both $H_a(P^2)$ $(= H_a(T^*T) = H_a((T^*T)^{1/2}))$ and $H_a(U)$ contain the least subspace, $M$, of $H$ reducing $P^2$ and $U$ (equivalently, reducing $P$ and $U$) and containing the range of $D = T^*T - TT^*$. Since $T$ is completely hyponormal, $M = H$, and, in particular, (6.1) follows.

7. **Theorem 7.** *Let $T$ be hyponormal and suppose that*

$$r \in \mathrm{sp}(T^*T) \quad (\text{hence } r \geq 0). \tag{7.1}$$

*Then there exists a $z \in \mathrm{sp}(T)$ for which $|z| = r^{1/2}$.*

**Remark.** The hypothesis (7.1) of Theorem 7 can be replaced by

$$r \in \mathrm{sp}(TT^*) \quad (\text{hence } r \geq 0). \tag{7.1$'$}$$

In fact, if $r = 0$ then $T$ is singular and $0 \in \mathrm{sp}(T)$. If $r > 0$, then (7.1)$'$ implies (7.1); cf. §1 above.

**Proof.** We first establish the theorem under the added hypothesis that $T$ has a polar factorization (1.1) and that $\mathrm{sp}(U)$ is not the entire circle $|z| = 1$. Thus,

$$T = UP \quad \text{and} \quad \mathrm{meas}_1(\mathrm{sp}(U)) < 2\pi. \tag{7.2}$$

Let $e^{i\theta} \notin \mathrm{sp}(U)$ and define the unitary operator $U_\theta = e^{-i\theta}U$. Then $1 \notin \mathrm{sp}(U_\theta)$ and relation (4.1) becomes $P^2 - U_\theta P^2 U_\theta^* = D$. Now, $U_\theta$ is the Cayley transform of a selfadjoint operator $A$, where

$$U_\theta = (A - iI)(A + iI)^{-1} \qquad (U_\theta = e^{-i\theta}U). \tag{7.3}$$

If $C = \frac{1}{2}(A + iI)D(A + iI)^*$, it is seen that

$$AP^2 - P^2A = iC, \qquad C \geq 0. \tag{7.4}$$

(For a similar argument, see [6, pp. 16, 21].)

Next, by (7.1), $r \in \mathrm{sp}(P^2)$, so that $(P^2 - rI)x_n \to 0$ for some sequence of unit vectors $\{x_n\}$. But, by (7.4), $i\|C^{1/2}x_n\|^2 = (AP^2x_n, x_n) - (Ax_n, P^2x_n) \to 0$ and so $Cx_n \to 0$, and hence by (7.4) again, $(P^2 - rI)Ax_n \to 0$. Similarly, one obtains

(7.5) $$(P^2 - rI)A^k x_n \to 0 \quad \text{as } n \to \infty \ (k = 0, 1, 2, \cdots).$$

In view of (7.3) and the relation $U_\theta^* = (A - iI)^{-1}(A + iI)$, it follows from (7.5) that

(7.6) $$(P^2 - rI)U^k x_n \to 0, \qquad n \to \infty \ (k = 0, \pm 1, \pm 2, \cdots).$$

Hence, by an argument similar to that of [6, p. 46] (see also §4 above) there exists some $e^{i\phi} \in \mathrm{sp}(U)$ and a sequence of unit vectors, $\{y_n\}$, such that $(P^2 - rI)y_n \to 0$ (hence $(P - r^{1/2}I)y_n \to 0$) and $(U - e^{i\phi}I)y_n \to 0$. Thus, if $z = r^{1/2}e^{i\phi}$ then $(T - zI)y_n \to 0$, thus $z \in \mathrm{sp}(T)$, and so Theorem 7 is proved in the special case in which (7.2) is assumed.

Next, we consider the general case of Theorem 7. Let $T$ have the rectangular representation $T = A + iB$ where $A$ has the spectral resolution $A = \int t\,dE_t$. Let $S_n = (-\infty, \infty) - (-1/n, 1/n)$ for $n = 1, 2, \cdots$ and consider the hyponormal operator $T_n = E(S_n)TE(S_n)$ defined on the Hilbert space $E(S_n)H$. Then $\mathrm{sp}(T_n)$ lies outside the strip $|\mathrm{Re}\,(z)| < 1/n$ (see [6, p. 46]) and also (see [8], [9])

(7.7) $$\mathrm{sp}\,(T_n) \subset \mathrm{sp}\,(T).$$

In particular, each $T_n$ is nonsingular and hence has a polar factorization $T_n = U_n P_n$ and, by Theorem 4, $\mathrm{meas}_1(\mathrm{sp}(U_n)) < 2\pi$ $(n = 1, 2, \cdots)$, so that (7.2) holds with the role of $T$ played by $T_n$. In addition, it is clear that

(7.8) $$T_n \to T, \qquad T_n^* \to T^* \quad \text{(strongly)} \ [T_n \text{ here as an operator on } H].$$

Consequently, $T_n^*T_n \to T^*T$ (strongly) and, by (7.1), there exist $r_n \in \mathrm{sp}(T_n^*T_n)$ for which $r_n \to r$. (The argument is similar to that following formula line (5.5) above.) Since the assertion of Theorem 7 has already been proved for the $T_n$, there exist $z_n \in \mathrm{sp}(T_n)$ such that $|z_n| = r_n^{1/2} \to r^{1/2}$. Since, by (7.7), $z_n \in \mathrm{sp}(T)$, and since $\{z_n\}$ is a bounded sequence, there exists a convergent subsequence $\{z_{n_k}\}$, say $z_{n_k} \to z$. Clearly, this $z$ satisfies the conditions of Theorem 7 and the proof is complete.

8. **Theorem 8.** *Let $T$ be completely hyponormal and suppose that*

(8.1) $$r\ (\geq 0) \text{ is an isolated point of } \mathrm{sp}\,(TT^*).$$

*Let $a = \inf\{s\colon s \in \mathrm{sp}(TT^*),\ s \leq r$ and, if $s < r$, $(s, r)$ contains no points of the essential spectrum of $TT^*\}$; and $b = \sup\{s\colon s \in \mathrm{sp}(TT^*),\ s \geq r$ and if $s > r$, $(r, s)$ contains no points of the essential spectrum of $TT^*\}$. Then $a < b$ and either*

(8.2) $$a < r \quad \text{and} \quad \{z\colon a^{1/2} < |z| < r^{1/2}\} \subset \mathrm{ptsp}\,(T^*)$$

*or*

(8.3) $$b > r \quad and \quad \{z: r^{1/2} < |z| < b^{1/2}\} \subset \text{ptsp}\,(T^*).$$

**Remark.** It should be noted that even if both inequalities $a < r < b$ hold, still, as simple examples show, only one of the relations (8.2) and (8.3) need hold.

**Proof.** First we show that $a < b$. Otherwise, $a = b = r$ and $\text{sp}(TT^*)$ is the singleton $\{r\}$. If $r = 0$ then $T = 0$, hence $T$ is normal, a contradiction. If $r > 0$ then, since $T^*T \geq TT^*$, $\text{sp}(T^*T) = \text{sp}(TT^*) = \{r\}$, that is, $T^*T = TT^* = rI$ and again $T$ must be normal, a contradiction.

It follows from Theorem 7 and the remark following it that there exists a number $z_0 \in \text{sp}(T)$ for which $|z_0| = r^{1/2}$. Also, there exist $z_n \in \text{sp}(T)$ with $|z_n| \neq |z_0|$ satisfying $z_n \to z_0$ as $n \to \infty$. Otherwise, there exists an open disk $\alpha$ centered at $z_0$ and such that $\alpha \cap \text{sp}(T)$ is not empty and has zero planar measure. This is impossible by (1.5). It follows from Theorem 1 and the definitions of $a$ and $b$ in Theorem 8 that no boundary points of $\text{sp}(T)$ can lie in the difference set $\{z: a^{1/2} < |z| < b^{1/2}\} - \{z: |z| = r^{1/2}\}$.

Since $|z_n| \neq |z_0| = r^{1/2}$ then, for any $n$, either $|z_n| < r^{1/2}$ or $|z_n| > r^{1/2}$. Suppose first that $|z_n| < r^{1/2}$ for some $n$. Then clearly $a < r$ and, since no boundary point of $\text{sp}(T)$ can lie in $\{z: a^{1/2} < |z| < r^{1/2}\}$, it follows that $\{z: a^{1/2} < |z| < r^{1/2}\} \subset \text{sp}(T)$. Relation (8.2) now follows from Theorem 2. Similarly, if $|z_n| > r^{1/2}$ for some $n$, relation (8.3) holds.

9. **Theorem 9.** *Let $T$ be completely hyponormal and suppose that*

(9.1) $$\text{meas}_1(\text{sp}\,(T^*T)) \quad (= \text{meas}_1(\text{sp}\,(TT^*))) = 0.$$

*Then there exists a finite or denumerably infinite number of pairwise disjoint open annuli $A_n = \{z: a_n < |z| < b_n\}$ $(n = 1, 2, \cdots)$ such that*

(9.2) $$\text{sp}\,(T) \textit{ is the closure of the set } \bigcup A_n$$

*and*

(9.3) $$\bigcup A_n \subset \text{ptsp}\,(T^*).$$

**Proof.** Let $z_0 \in \text{sp}(T)$. Then consider any open disk $\alpha$ containing $z_0$. Then necessarily $\alpha$ contains a closed disk $\beta$ satisfying

(9.4) $$\beta = \{z: |z - z_1| \leq s,\ s > 0\} \subset \text{sp}\,(T).$$

In fact, otherwise, all points of $\alpha \cap \text{sp}(T)$ would be boundary points of $\text{sp}(T)$. Further, if the half-line $L: \theta = c$ (const.) intersects $\alpha$, then, by Theorem 1, each $r \geq 0$ satisfying $re^{ic} \in L \cap (\alpha \cap \text{sp}(T))$ belongs to $\text{sp}\,(T^*T)^{1/2}$. Hence, by (9.1), the set of such numbers $r$ has linear measure 0. It readily follows from Fubini's theorem that $\alpha \cap \text{sp}(T)$ (which contains $z_0$ and hence is not empty) has zero planar measure and hence, by (1.5), $T$ is not completely hyponormal, a contradiction. This proves (9.4).

If $a = \inf\{|z|: z \in \beta\}$ and $b = \sup\{|z|: z \in \beta\}$ then $a < b$. By (9.1), $\mathrm{sp}(T^*T)^{1/2}$ cannot contain $[a, b]$ and it follows from Theorem 1 that

(9.5) $$\{z: a \leq |z| \leq b\} \subset \mathrm{sp}(T).$$

It is clear then that $\mathrm{sp}(T)$ is the closure of a set consisting of a possibly uncountable number of closed annuli each of the form (9.5). By a standard procedure (of combining intersecting annuli), one easily shows that $\mathrm{sp}(T)$ can be taken as the closure of a countable union of disjoint closed annuli, each of the form $\{z: c \leq |z| \leq d\}$ with $c < d$.

For a fixed such annulus let $\bigcup (c_n, d_n)$ denote the canonical decomposition of the linear open set $[c, d] - \{[c, d] \cap \mathrm{sp}(TT^*)^{1/2}\}$. (Note that any $z$ satisfying $|z| = d$ is a boundary point of $\mathrm{sp}(T)$ and that a similar statement holds for $|z| = c$ provided $c > 0$. In view of $T^*T \geq TT^*$ it is clear from Theorem 1 that both $c$ and $d$ (even if $c = 0$) belong to $\mathrm{sp}(TT^*)^{1/2}$.) It follows from (9.1) that $\{z: c \leq |z| \leq d\}$ is the closure of $\bigcup B_n$, where $B_n = \{z: c_n < |z| < d_n\}$, and, from Theorem 2, that each $B_n \subset \mathrm{ptsp}(T^*)$. This completes the proof of Theorem 9.

A final result is the following

**Theorem 10.** *Let $T$ be hyponormal and suppose that*

(9.6) $$\mathrm{sp}(TT^*) \neq \textit{interval}.$$

*Then $T$ has a nontrivial invariant subspace.*

**Proof.** Clearly, it can be supposed that $T$ is completely hyponormal. Further, by (9.6) (cf. the beginning of the proof of Theorem 8), $\mathrm{sp}(TT^*)$ contains at least two points $r_1$ and $r_2$ satisfying $r_1 < r_2$ and for which $(r_1, r_2) \cap \mathrm{sp}(TT^*)$ is empty. It follows from Theorem 7 and the remark following it that there exist $z_1, z_2 \in \mathrm{sp}(T)$ where $|z_1| = r_1^{1/2}$ and $|z_2| = r_2^{1/2}$.

Clearly, $T$ has a nontrivial invariant subspace if $\mathrm{ptsp}(T^*)$ is not empty. Hence, it can be supposed that this set is empty, and so, by Theorem 2, $\mathrm{sp}(T) \cap \{z: r_1^{1/2} < |z| < r_2^{1/2}\}$ is empty. Thus, $\mathrm{sp}(T)$ is not connected and hence (cf. [10, p. 421]) $T$ has a nontrivial invariant subspace.

It may be noted that Theorem 10 is applicable to the unilateral shift but that it would not be if (9.6) is replaced by the (stronger) condition $\mathrm{sp}(T^*T) \neq$ interval.

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